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Two massive and one massless Sp(4) monopole

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Starting from Nahm's equations, we explore Bogomol'nyi-Prasad-Sommerfield (BPS) magnetic monopoles in the Yang-Mills-Higgs theory of the gauge group Sp(4), which is broken to SU(2)×U(1). There exists a family of BPS field configurations with a purely Abelian magnetic charge that describes two identical massive monopoles and one massless monopole. We construct the field configurations with axial symmetry by employing the Atiyah-Drinfeld-Hitchin-Mannin-Nahm construction and find the explicit expression of the metrics for the 12-dimensional moduli space of Nahm data and its submanifolds. [S0556-2821(98)01408-8]

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I. INTRODUCTION

In this paper we consider the Yang-Mills-Higgs theory whose gauge symmetry Sp(4) is broken to SU(2)×U(1), where the Higgs field expectation value lies along one of the short roots. We investigate a family of purely Abelian configurations that describes two identical massive monopoles and one massless monopole. We approach the problem by solving Nahm's equations under proper boundary and compatibility conditions. By using the Atiyah-Drinfeld-Hitchin-Mannin-Nahm (ADHMN) construction [1,2], we construct the field configurations in spherically and axially symmetric cases. We then calculate the metrics of the 12-dimensional moduli space M^{12} of Nahm data and its submanifolds. Generally, it is expected that the moduli space of Nahm data is isometric to the moduli space of the corresponding monopole configurations. We examine the metric of the moduli space in detail and show that it behaves consistently with what is expected from the dynamics of monopoles.

Recently, magnetic monopoles have again become a focus of attention as they play a crucial role in the study of electromagnetic duality in the supersymmetric Yang-Mills theories. The relevant magnetic monopole solutions are of the Bogomol'nyi-Prasad-Sommerfield (BPS) type [3]. The gauge inequivalent field configurations of the BPS monopole solutions are characterized by the moduli parameters associated with the zero modes of the solutions. The metric of the moduli space determines the low-energy dynamics of monopoles [4]. The electromagnetic duality has been explored by studying quantum mechanics on the moduli space of the BPS monopoles.

When the gauge group is not maximally broken so that there is an unbroken non-Abelian gauge symmetry, the moduli space dynamics becomes more subtle because of the global color problem [5]. Nevertheless, it has been known that the moduli space is well defined when the total magnetic charge is purely Abelian [6]. Recently, some such moduli spaces have been studied by starting from the maximal symmetry-breaking case and restoring the broken symmetry partially [7]. From this point of view some magnetic monopoles become massless, forming a non-Abelian cloud surrounding remaining massive monopoles. The global part of the unbroken gauge symmetry becomes the isometry of the moduli space. The meaning of the moduli space coordinates of massless monopoles changes from their positions and phases to the gauge-invariant cloud structure parameters and the gauge orbit parameters. With an inequivalent symmetry breaking Sp(4)→SU(2)×U(1) with the Higgs expectation along a long root, an Abelian combination is made of one massive monopole and one massless monopole. This simple case, where the field configuration and the moduli space metric are completely known, was studied in detail to learn about the non-Abelian cloud [7,8].

The next nontrivial purely Abelian configurations beyond this simple model are made of two massive monopoles and one massless monopole. Two massive monopoles can be distinguished as in the example where SU(4)→U(1)×SU(2)×U(1). In that case, the so-called Taubian-Calabi metric for the moduli space [7,9,10] is obtained from the massless limit of that of the maximally broken case [11]. Two massive monopoles are identical in the cases where SU(3), Sp(4), and G_2→SU(2)×U(1). (See Tables I and II of Ref. [7].) Some time ago the moduli space of three monopoles in the theory where SU(3)→SU(2)×U(1) has been found by Dancer by exploring the moduli space of Nahm's data [12,13].

Our approach is similar to Dancer's. We use the embedding procedure to construct Sp(4) configurations from SU(4) configurations. Some of the field configurations are simpler than Dancer's. Our spherical symmetric solution is just an embedding of the SU(2) solution. A class of our axially symmetric solutions can be obtained from a linear superposition of configurations for two noninteracting monopoles. Our work provides a further illustration of the role of massless monopoles.

Another motivation for studying the moduli space of configurations involving massless monopoles is that it may lead us to some further insight about mesons and baryons in quenched QCD. Even in quenched QCD, nondynamical external quarks are expected to be confined and form mesons and baryons. Suppose that quenched QCD were supersymmetrized to $N=4$ so that there is no confinement. (Here we imagine that all supersymmetric partners are very massive initially and then become light.) If the coupling constant is

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still strong, the resulting configurations of mesons and baryons cannot be described by Coulomb potentials as the nonlinear gauge interaction is not negligible. The non-Abelian gauge field should somehow form a cloud around external quarks, making the whole configuration to be a gauge singlet because of the continuity of the configuration with respect to coupling parameters. The shape of this cloud, which can be regarded as a tensionless cloud, is reminiscent of confinement strings that connected the quarks. This can be regarded as the limit where confining string becomes tensionless.

If the electromagnetic duality holds even when the unbroken gauge symmetry is partially non-Abelian [14], mesons and baryons can have their magnetic dual, which are made of massive and massless monopoles. Indeed massive monopoles play the role of external quarks and massless monopoles play that of non-Abelian cloud. Thus Abelian configurations made of two massive monopoles and one massless monopole can be regarded as dual mesons. More interestingly, the moduli space of three massive and three massless monopoles in the theory of $SU(4)\to SU(3)\times U(1)$ can be regarded as a magnetic dual of baryons [15]. The structure of dual baryons may be similar to a shape of confinement strings connecting three external quarks.

The plan of this work is as follows. In Sec. II we review the method to find Nahm data for the classical group. In Sec. III we study the symmetry-breaking pattern $Sp(4)\to SU(2)\times U(1)$ and solve Nahm's equations with relevant boundary conditions. In Sec. IV we use the ADHMN method to construct the Higgs field configurations in spherically and axially symmetric cases. This leads to a general understanding of the parameter space in terms of the size of non-Abelian cloud and the distance between massive monopoles. In Sec. V we find the explicit metrics of the moduli space and its submanifolds. In Sec. VI we conclude with some remarks.

II. NAHM DATA

The Bogomol'nyi equations satisfied by BPS monopoles can be written as self-dual Yang-Mills equations

$$F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{1}$$

in R^4 with coordinates x_1,x_2,x_3,x_4. All the fields of BPS monopoles here depend only on x_1,x_2,x_3. Instead, if everything depends only on the complementary variable $x_4=t$, then Eq. (1) leads to the so-called Nahm equations

$$\frac{dA_i}{dt}+[A_4,A_i]=\frac{1}{2}\epsilon_{ijk}[A_j,A_k], \tag{2}$$

where $i,j,k=1,2,3$. The solutions of Nahm's equations satisfying certain boundary conditions are called Nahm data. We can always perform a gauge transformation to eliminate A_4, so sometimes A_4 is not included in Nahm's equations. Nahm's equations are much easier to solve than the original self-dual Yang-Mills equations since they are ordinary differential equations. The relationship between Bogomol'nyi equations (depend on three variables) and Nahm's equations has been thoroughly investigated especially in the SU(2) gauge group case [2,16]. There is a kind of duality between d- and $(4-d)$-dimensional self-dual theories [17]. It is also believed in general that the moduli spaces of Nahm data and BPS monopoles are isometric to each other, which has been proved in the SU(2) case [18]. The idea is that Nahm's equations are regarded as an infinite-dimensional moment map and that the hyperkähler quotient [19] of the infinite-dimensional flat space will lead to the natural hyperkähler metric for the moduli space of Nahm data [18,20].

The original Nahm method of SU(2) monopoles has been generalized into all types of classical groups [2,21]. Let us start with the $SU(N)$ case since all other groups can be treated by embedding them into $SU(N)$. Assuming that the asymptotic Higgs field is $\phi_\infty=\mathrm{diag}(\mu_1,\ldots,\mu_N)$, with $\mu_1<\cdots<\mu_N$ along a given direction, then the Nahm data for multi-monopoles carrying charge $(m_1,\ldots,m_{N-1})$ are defined as $N-1$ triples $({}^lT_1,{}^lT_2,{}^lT_3)$ $(l=1,\ldots,N-1)$ satisfying the following.

(i) For each l, ${}^lT_i$ $(i=1,2,3)$ are analytic $u(m_l)$-valued functions satisfying Nahm's equations in interval (μ_l,μ_{l+1}), $l=1,\ldots,N-1$.

(ii) The boundary conditions relating the Nahm data in two adjoint intervals are the following.

(a) If $m_l>m_{l-1}$, then there exists a nonsingular limit $\lim_{t\to\mu_l^-}{}^{l-1}T_i={}^{l-1}S_i$ and the structure of ${}^lT_i$ near $t=\mu_l$ is

$$\lim_{t\to\mu_l^+}{}^lT_i=\begin{pmatrix} {}^{l-1}S_i & * \\ * & \dfrac{{}^lR_i}{t-\mu_l}\end{pmatrix}, \tag{3}$$

where ${}^lR_i$ form an (m_l-m_{l-1})-dimensional irreducible representation of SU(2) [unless $m_l-m_{l-1}=1$, in which case ${}^lR_i/(t-\mu_l)$ has to be replaced by a nonsingular expression] and the asterisks refer to the elements that are not interesting in this paper.

(b) If $m_l<m_{l-1}$, the roles of (μ_{l-1},μ_l) and (μ_l,μ_{l+1}) are reversed.

(c) If $m_l=m_{l-1}$, the condition is more complicated, but fortunately we are not going to confront this situation in this paper.

The way to embed the cases of $SO(N)$ and $Sp(N)$ into the $SU(N)$ group is described in Table I [21]. These embedding procedures are obtained by constraining the $SU(N)$ generators further. The generators T of $Sp(N)$ satisfy the condition $T^TJ+JT=0$ such that $JJ^*=-I$. The generators T of $SO(N)$ satisfy the condition $T^TK+KT=0$ such that $KK^*=I$. The explicit forms of J,K can be deduced from Table I.

These embedding procedures enable us to get the $SO(N)$ and $Sp(N)$ Nahm data from the $SU(N)$ data with asymptotic Higgs field $\phi_\infty=\mathrm{diag}(\mu_1,\ldots,\mu_N)$ and the charge $\{m_l\}$. What is different is that we now have one more set of conditions connecting the Nahm data between different intervals.

(iii) There exist matrices ${}^lC$ $(l=1,\ldots,N-1)$ satisfying

$${}^{N-l}T_i(-t)^T=({}^lC)\,{}^lT_i(t)\,({}^lC^{-1}) \tag{4}$$

and compatibility conditions (a) ${}^{N-l}C={}^lC^T$ for $Sp(N)$ and (b) ${}^{N-l}C=-{}^lC^T$ for $SO(N)$. These compatibility conditions reflect the fact that we are identifying certain $SU(N)$ monopoles to get $SO(N)$ and $Sp(N)$ monopoles.

TABLE I. Embedding of Sp(N), SO(N), and SU(N).

G	G charge	ϕ_∞ in SU(N)	SU(N) charge
Sp(N)	$\rho_1,\ldots,\rho_n$	$\mu_l=-\mu_{2n+1-l}$	$m_l=m_{2n-l}=\rho_l$
$N=2n$		$l=1,\ldots,n$	$l=1,\ldots,n$
SO(N)	$\rho_1,\ldots,\rho_{n-2}$	$\mu_l=-\mu_{2n+1-l}$	$m_l=m_{2n-l}=\rho_l$
$N=2n$	ρ_+,ρ_-	$l=1,\ldots,n$	$l=1,\ldots,n-2$
			$m_{n-1}=m_{n+1}=\rho_++\rho_-$
			$m_n=2\rho_+$
SO(N)	$\rho_1,\ldots,\rho_n$	$\mu_l=-\mu_{2n+2-l}$	$m_l=m_{2n+1-l}=\rho_l$
$N=2n+1$		$l=1,\ldots,n+1$	$l=1,\ldots,n-1$
			$m_n=m_{n+1}=2\rho_n$

In the above discussion we have assumed that $\mu_1<\cdots<\mu_n$, which physically means that the gauge symmetry is maximally broken. We can also consider the cases with non-Abelian unbroken symmetry so that some μ_l's are equal; geometrically this is the case when some of the intervals shrink to zero length. The monopole mass is proportional to the size of the corresponding interval and so the shrunken intervals correspond to massless monopoles. All the procedures described above remain unchanged even in this case.

III. NAHM DATA IN THE Sp(4) CASE

The model we consider is the Sp(4) Yang-Mills theory with a single Higgs field in the adjoint representation and no potential. The vacuum expectation value of the Higgs field is nonzero and the gauge symmetry is spontaneously broken to $SU(2)\times U(1)$. The roots and coroots of the Sp(4)=SO(5) group are shown in Fig. 1. Note that in our convention $\alpha^*=\alpha/|\alpha|^2=\alpha$.

In this paper we consider the symmetry breaking with $\langle\Phi\rangle=h\cdot H$ along a short root γ. The simple roots we choose for convenience are β,α rather than $\delta,-\alpha$. For any root α, there is a corresponding SU(2) subalgebra

$$t^1(\alpha)=\frac{1}{\sqrt{2\alpha^2}}(E_\alpha+E_{-\alpha}),$$

$$t^2(\alpha)=\frac{-i}{\sqrt{2\alpha^2}}(E_\alpha-E_{-\alpha}),$$

$$t^3(\alpha)=\alpha^*\cdot H. \tag{5}$$

Using this SU(2) algebra, we can embed the SU(2) single monopole solution along any root. Thus there is a spherically symmetric monopole configuration for any root α such that $\alpha\cdot h\neq 0$ [22]. Since $\beta\cdot h>0$, the monopole with magnetic charge β^* is massive. (Here we are dropping the coupling constant $4\pi/e$.) On the other hand, $\alpha^*\cdot h=0$ and so there is no monopole solution corresponding to the root α. As argued in the Introduction, the zero mode counting can be done consistently only for purely Abelian configurations. In our case the simplest case has the magnetic charge

$$\gamma^*=2\beta^*+\alpha^*, \tag{6}$$

so that $\gamma^*\cdot\alpha=0$. The moduli space of this configuration is 12 dimensional and denoted by M^{12}. As discussed in Ref. [7], we imagine the h as a limit where $h\cdot\alpha$ is positive but becomes infinitesimal. We can regard α^* monopoles as massless and so the γ^* monopole can be thought of as a composite of two identical massive β^* monopoles and one massless α^* monopole. Here we can see that the internal unbroken gauge group should be $SO(3)_g$ rather than SU(2) because all the generators of Sp(4) transforms as vector or singlet representations under the unbroken generators $\mathbf{t}(\alpha)$.

If we have chosen the Higgs expectation value to be h' in Fig. 1, the unbroken SU(2) would be associated with β. The Abelian configuration could have the magnetic charge $\delta^*=\alpha^*+\beta^*$ so that $\delta^*\cdot\beta=0$. This configuration can be interpreted as a composite of one massive α^* monopole and one massless β^* monopole. The BPS field configuration and eight-dimensional moduli space of this magnetic charge are known explicitly to be flat R^4. This is the model that has led to many insights into non-Abelian cloud [7].

As discussed in Sec. II, Nahm data for Sp(4) can be studied by embedding Sp(4) in SU(4). Thus the Higgs field can be written as a 4×4 traceless Hermitian matrix. As shown in Table I, the Higgs expectation value can be chosen to be $\langle\Phi\rangle=\mathrm{diag}(-\mu_1,-\mu_2,\mu_2,\mu_1)$ with $\mu_1\geqslant\mu_2\geqslant 0$. Any generator T of the Sp(4) subgroup should be traceless anti-Hermitian and satisfy

$$TJ+JT^T=0, \tag{7}$$

where the Sp(4) invariant tensor J is chosen to be

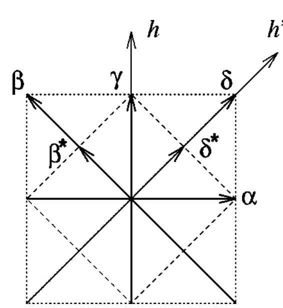

FIG. 1. Root diagram of Sp(4).

$$J=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

This defines the Sp(4) embedding in SU(4) uniquely, which is also consistent with Table I. A consistent choice of the Cartan subgroup of Sp(4) is $H_1=\mathrm{diag}(-1,1,-1,1)/2$ and $H_2=\mathrm{diag}(-1,-1,1,1)/2$. The two inequivalent symmetry-breaking patterns for $\mathrm{Sp}(4)\to\mathrm{SU}(2)\times\mathrm{U}(1)$ in Fig. 1 correspond to $h\cdot H=\mathrm{diag}(-1,-1,1,1)$ and $h'\cdot H=(-1,0,0,1)=H_1+H_2$. Thus our case with $\mu_1=\mu_2=0$ corresponds to the case where $\mathrm{SU}(4)\to\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{SU}(2)$.

From Table I in Sec. II, we read that our configuration (6) in Sp(4) has the SU(4) magnetic charge $(1,2,1)$, that is, two identical massive monopoles and two distinct massless monopoles. This is exactly the configuration considered by Houghton [23], whose focus was on its hyperkähler quotient spaces. If we have chosen the expectation value h', the simplest Abelian configurations have the magnetic charge $(1,1,1)$ in SU(4), that is, two distinct massive monopoles and one massless monopole, whose moduli space metric has been found to be the Taubian-Calabi metric [7,9,10].

According to Sec. II, Nahm data $T_\mu(t)$ defined on the interval $[-1,1]$ are anti-Hermitian 2×2 matrices and satisfy Nahm's equations

$$\frac{dT_i}{dt}+[T_4,T_i]=\frac{1}{2}\epsilon_{ijk}[T_j,T_k] \tag{9}$$

and the compatibility condition

$$T_\mu(-t)^T=CT_\mu(t)C^{-1} \tag{10}$$

with a symmetric matrix C. The Nahm data should be analytic at the end points $t=\pm 1$. The boundary and compatibility conditions (3) and (4) satisfied by the above Nahm data become

$$[T_\mu(-1)]_{22}=[T_\mu(1)]_{22}. \tag{11}$$

This boundary value of Nahm data can be identified with the position of the massless monopole in the center-of-mass frame. A detailed understanding of this boundary condition will be needed in the case where α^* monopoles become massive.

The space of Nahm data has the following symmetries: (i) local gauge transformations $\mathcal{G}=\{g(t)\in\mathrm{U}(2)\}$, whose transformations are

$$T_4\to gT_4g^{-1}-\frac{dg}{dt}g^{-1},$$

$$T_i\to gT_ig^{-1}, \tag{12}$$

which should be consistent with the conditions (10) and (11), and whose subgroup is $\mathcal{G}_0=\{g\in\mathcal{G}: g(-1)=g(1)=1\}$; (ii) the spatial translation group R^3 with three parameters λ_i,

$$T_4\to T_4,$$

$$T_j\to T_j-i\lambda_jI; \tag{13}$$

and (iii) the spatial rotation group $\mathrm{Sp}(3)=\{a_{ij}\in\mathrm{SO}(3)\}$,

$$T_i\to\sum_j a_{ij}T_j. \tag{14}$$

Notice that Eq. (14) is a pure rotation as there is no residue to be fixed at $t=\pm 1$. [This indicates that the rotational group is SO(3) rather than SU(2).]

To solve Nahm's equations together with the compatibility condition, we use the spatial translation to make T_μ traceless. These traceless Nahm data are called centered and describe monopole configurations in the center-of-mass frame. We can also choose the gauge $T_4=0$. Furthermore, we use a spatial rotation to set the t-independent tr (T_1T_2), tr (T_1T_3), and tr (T_2T_3) to be zero. After a gauge rotation, we get that, for each $j=1,2,3$,

$$T_j=\frac{1}{2}f_je_j, \tag{15}$$

where quaternions e_j are chosen so that

$$e_1=i\tau_1,\quad e_2=i\tau_3,\quad e_3=i\tau_2, \tag{16}$$

with Pauli matrices τ_j. Then Nahm's equations become the well-known Euler top equations

$$\dot f_1=f_2f_3,$$

$$\dot f_2=f_3f_1,$$

$$\dot f_3=f_1f_2. \tag{17}$$

We note that $f_1^2-f_2^2$ and $f_2^2-f_3^2$ are independent of t. Hence let us consider the case $f_1^2\leqslant f_2^2\leqslant f_3^2$. Then the solution to this set of equations is known in terms of Jacobi elliptic functions as

$$f_1=-\frac{D\ \mathrm{cn}_k[D(t-t_0)]}{\mathrm{sn}_k[D(t-t_0)]},$$

$$f_2=-\frac{D\ \mathrm{dn}_k[D(t-t_0)]}{\mathrm{sn}_k[D(t-t_0)]},$$

$$f_3=-\frac{D}{\mathrm{sn}_k[D(t-t_0)]}, \tag{18}$$

where $k\in[0,1]$ is the elliptic modulus and D,t_0 are arbitrary. We can change the sign of any two of f_1, f_2, and f_3 by 180° spatial rotations.

On the other hand, the compatibility condition (10) becomes, for every j,

$$f_j(-t)\tau_j^T=f_j(t)C\tau_jC^{-1}, \tag{19}$$

with a symmetric matrix C. The boundary condition (11) becomes $f_2(-1)=f_2(1)$. Among linear combinations of τ_1 and τ_3, the right choice for C with Nahm data (15) is

$$C = \tau_3 . \tag{20}$$

This implies that f_1 is an odd function and f_2, f_3 are even functions.[1] This fixes the parameter t_0 to satisfy $\mathrm{cn}_k(Dt_0) = 0$. Then our solution for Nahm's equation is

$$f_1 = D\sqrt{1-k^2}\frac{\mathrm{sn}_k(Dt)}{\mathrm{cn}_k(Dt)},$$

$$f_2 = -D\sqrt{1-k^2}\frac{1}{\mathrm{cn}_k(Dt)},$$

$$f_3 = -D\frac{\mathrm{dn}_k(Dt)}{\mathrm{cn}_k(Dt)}. \tag{21}$$

These Nahm data are regular for $t \in [-1,1]$. The analyticity of the data requires that $0 \leqslant k \leqslant 1$ and $0 \leqslant D \leqslant K(k)$ with $4K(k)$ being the period of sn_k. $K(k)$ is also the first complete elliptic integral $K = \int_0^{\pi/2} d\theta (1-k^2\sin^2\theta)^{-1/2}$. Equations (15) and (21) are the Nahm data we are looking for. [Actually they are the Nahm data on a representative point of the $\mathrm{SO}(3)\times\mathrm{SO}(3)$ orbit.] Sometimes we will simply call Eq. (21) the Nahm data. There are eight equivalent copies of the above Nahm data: We can exchange f_2 and f_3 or change the signs of any two of f_1, f_2, and f_3. The allowed local gauge transformations of Eq. (12) are made of $g(t)$ such that

$$g(t) = e^{\epsilon_j(t) e_j/2} \tag{22}$$

with even $\boldsymbol{\epsilon}_1$ and odd $\boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3$ functions. This will be crucial in showing that the spherically symmetric Nahm data are not invariant under global gauge transformations due to $\boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3$.

The moduli space M^{12} of uncentered three monopoles is the space of gauge inequivalent Nahm data with the gauge action $\mathcal{G}_0$. Since the center U(1) of U(2) is triholomorphic, we can perform a hyperkähler quotient with the momentum map $\mu = -i(\mathrm{tr}\, T_1, \mathrm{tr}\, T_2, \mathrm{tr}\, T_3)$. This gives the eight-dimensional relative modulus space M^8 of the centered Nahm data. A further quotient of this manifold by the internal gauge symmetry SU(2) leads to the five-dimensional manifold $N^5 = M^8/\mathrm{SU}(2)$. The homeomorphic coordinates for N^5 are given in terms of gauge-invariant t-independent quantities [12]

$$\lambda_1 = \langle T_1, T_1\rangle - \langle T_2, T_2\rangle,$$

$$\lambda_2 = \langle T_1, T_1\rangle - \langle T_3, T_3\rangle,$$

$$\lambda_3 = \langle T_1, T_2\rangle,$$

$$\lambda_4 = \langle T_1, T_3\rangle,$$

$$\lambda_5 = \langle T_2, T_3\rangle, \tag{23}$$

where

$$\langle T, T'\rangle = -\int_{-1}^{1} dt\ \mathrm{tr}\,(TT'). \tag{24}$$

They form a real traceless 3×3 matrix and realize a five-dimensional representation of SO(3). The data (21) lead to the coordinates

$$\lambda_1 = -(1-k^2)D^2,$$

$$\lambda_2 = -D^2, \tag{25}$$

and $\lambda_3 = \lambda_4 = \lambda_5 = 0$, which is invariant under the 180° rotations around three Cartesian axes. Thus these data have a $Z_2 \times Z_2$ isotropy group. N^5 is a five-dimensional manifold homeomorphic to R^5 and admits a nonfree rotational SO(3) action. A further quotient of this manifold by the spatial rotation group SO(3) leads to a two-dimensional surface $N^5/\mathrm{SO}(3)$, whose eight copies, as we will argue in Sec. V, make a geodesic complete manifold $\mathcal{Y}^2$. There are also two-dimensional surfaces of revolution, which describe axially symmetric configurations.

Since the gauge group SU(2) is triholomorphic, there is another hyperkähler quotient of M^8. When the gauge symmetry is maximally broken, there is still an unbroken U(1) in the center-of-mass frame. We can use this U(1) to construct the hyperkähler quotient. With our choice of the boundary condition (11) and quaternions (16), a convenient choice of this U(1) subgroup is one generated by τ_3. (Other choices are gauge equivalent to this choice.) This U(1) group acts freely as the corresponding gauge parameter $\boldsymbol{\epsilon}_2$ in Eq. (22) is an odd function. The corresponding moment map is

$$\mu = i[\mathrm{tr}\ T_1(1)\tau_3], \mathrm{tr}\ [T_2(1)\tau_3], \mathrm{tr}\ [T_3(1)\tau_3]. \tag{26}$$

The value ζ_j of this moment map is then

$$\zeta_j = -i[T_j(1)]_{22} \tag{27}$$

and can be interpreted as the position of the massless monopole. The hyperkähler quotient space $M^4(\boldsymbol{\zeta}) = \mu^{-1}(\boldsymbol{\zeta})/\mathrm{U}(1)$ is a four-dimensional hyperkähler space. The rotational transformation $\mathrm{SO}(3) = \{a_{ij}\}$ generates a homeomorphic mapping from $M^4(\zeta_i)$ to $M^4(a_{ij}\zeta_j)$. (Under a gauge transformation of Nahm data, the moment map transforms nontrivially. The gauge orbit of the position $\boldsymbol{\zeta}$ of the massless monopole will be shown to be an ellipsoid.) This family will be shown to interpolate the flat space $M^4(0) = R^3 \times S^1$ to the Atiyah-Hitchin space $M^4(\infty)$. Since any hyperkähler space in four dimensions is self-dual and so Ricci flat, $M^4(\boldsymbol{\zeta})$ can be regarded as a one-parameter family of gravitational instantons.

IV. ADHMN CONSTRUCTION

For given Nahm data, we can define a differential operator

$$\Lambda^\dagger(\mathbf{x}) = i\frac{d}{dt} - \sum_{i=1}^{3} (iT_j + x_j) \otimes e_j, \tag{28}$$

where e_j $(j=1,2,3)$ are quaternion units. The dimension of the kernel of $\Lambda^\dagger$ depends on the boundary conditions in-

[1] Thus the generic form of the Nahm data would be given by $T_\mu = \mathcal{T}_{\mu\nu} e_\nu$ with $e_4 = 1$. $\mathcal{T}_{\mu 4}$ are independent of t, $\mathcal{T}_{\mu 1}$ are odd functions of t, and $\mathcal{T}_{\mu 2}$ and $\mathcal{T}_{\mu 3}$ are even functions.

volved in defining Nahm data T_i. For our case it turns out to be four. The basis of Ker $\Lambda^\dagger$ consists of four orthonormal four-component vectors $\mathbf{v}_\mu, \mu=1,\cdots,4$ with the inner products $\langle \mathbf{v}_\mu, \mathbf{v}_\nu\rangle = \int_{-1}^{1} dt\ \mathbf{v}_\mu^\dagger \cdot \mathbf{v}_\nu = \delta_{\mu\nu}$. In terms of the 4×4 matrix $V=(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)$, the ADHMN construction of monopole solutions in R^3 goes as follows: The 4×4 Hermitian matrix-valued fields

$$\Phi = \int_{-1}^{1} dt\ tV^\dagger V, \tag{29}$$

$$A_j = i\int_{-1}^{1} dt\ V^\dagger \frac{\partial V}{\partial x_j} \tag{30}$$

form a BPS monopole field configuration. It is really a configuration in SU(4) gauge theory and may need a further gauge transformation in SU(4) to be expressed as a proper Sp(4) configuration.

We express a single four vector as $\mathbf{v}=(w_1,w_2,w_3,w_4)^T$. Since the ADHMN construction is invariant under constant gauge transformations of Nahm data, we can rotate e_1,e_2,e_3 to be e_2,e_3,e_1, respectively. Then the equation $\Lambda^\dagger \mathbf{v}=0$ can be written in the same form as those in Ref. [13]:

$$\dot{w}_1 - x_1 w_1 - (x_3 - ix_2)w_3 + \frac{1}{2}f_1 w_1 + \frac{1}{2}(f_3 - f_2)w_4 = 0,$$

$$\dot{w}_2 - x_1 w_2 - (x_3 - ix_2)w_4 - \frac{1}{2}f_1 w_2 + \frac{1}{2}(f_2 + f_3)w_3 = 0,$$

$$\dot{w}_3 + x_1 w_3 - (x_3 + ix_2)w_1 - \frac{1}{2}f_1 w_3 + \frac{1}{2}(f_2 + f_3)w_2 = 0,$$

$$\dot{w}_4 + x_1 w_4 - (x_3 + ix_2)w_2 + \frac{1}{2}f_1 w_4 + \frac{1}{2}(f_3 - f_2)w_1 = 0. \tag{31}$$

It is hard to obtain general solutions of the above equations. In this section we would like to work out several special cases in order to check whether the ADHMN construction leads to the sensible result. This exercise also yields a general understanding of the physical meaning of parameters k and D appearing in Nahm data.

The first case we consider is the spherically symmetric solution with $D=0$ and so

$$f_1 = f_2 = f_3 = 0. \tag{32}$$

Clearly these Nahm data are invariant under the spatial SO(3) rotation (14). One may wonder whether these Nahm data are invariant under global gauge transformations (12). The above data $T_i=0$, are invariant under the global SO(3) gauge rotation (12). However, the initial $T_4=0$ is not necessarily invariant. The reason is that the gauge parameters ϵ_2,ϵ_2 of Eq. (22) are odd functions and so their time derivative does not vanish. However, ϵ_1 is even and so can be constant, leaving T_0 invariant. Thus one expects a S^2 gauge orbit space for the spherically symmetric solution. This two-sphere will also appear in the metric of the modulus space in Sec. V. [In Dancer's case, the spherically symmetric solution is not invariant for all three generators of the SU(2) gauge rotation and so the gauge orbit is S^3.]

The kernel equations (31) can be easily solved for the spherically symmetric solution and give rise to the Higgs field

$$\Phi = 2H(2r)\ \hat{\mathbf{r}}\cdot \mathbf{t}(\gamma), \tag{33}$$

where $r=\sqrt{x_i x_i}$, $\hat{r}_i = x_i/r$, and $H(r)=\coth(r)-1/r$. This is the well-known single-monopole solution with $\Phi_\infty \propto H_2$ along the x_3 direction. This configuration is the SU(2) embedded solution along the composite root γ. The energy density is maximized at the center. We just argued that the corresponding Nahm data are not invariant under some of the global gauge transformations. To understand this in terms of the field configuration, we deduce from the root diagram in Fig. 1 that the generators $t^i(\gamma)$ commute with $t^3(\alpha)$, but not with $t^{1,2}(\alpha)$. Thus the spherically symmetric field configuration is not invariant under two of $t^i(\alpha)$, consistent with the previous argument.

We now turn to the axially symmetric cases. Similar to Dancer's case, we have two axially symmetric cases. The hyperbolic case appears when $k=1$ and $0\leqslant D<\infty$, so that

$$f_1 = f_2 = 0, \quad f_3 = D. \tag{34}$$

These Nahm data are invariant under rotation around the x_3 axis. Although no hyperbolic function is involved here, we have used the same terminology as used as in Ref. [12] because of a similarity in their qualitative behavior. The trigonometric case appears when $k=0$, so that

$$f_1 = D\tan(Dt), \quad f_2 = f_3 = -D\sec(Dt), \tag{35}$$

with $0\leqslant D</\pi/2$. These data are invariant under the rotation around the x_1 axis.

Our hyperbolic case is much simpler than the corresponding case considered by Dancer. After solving Eq. (31), we use Eq. (29) and a gauge transformation to obtain the Higgs configuration

$$\Phi = 2H(2r_+)\hat{\mathbf{r}}_+\cdot \mathbf{t}(\beta) + 2H(2r_-)\hat{\mathbf{r}}_-\cdot \mathbf{t}(\delta), \tag{36}$$

where $\mathbf{r}_\pm = (x_1,x_2,x_3\pm D/2)$. We recognize that this configuration describes β^* and δ^* monopoles located at $x_3 = -D/2$ and $x_3=D/2$, respectively. Since $[t^i(\beta),t^j(\delta)]=0$, there is no direct interaction between these two monopoles and the field configuration (36) is just a superposition of two corresponding configurations. In Dancer's hyperbolic case, two massive monopoles are interacting.

The above hyperbolic configuration is not invariant under global gauge rotations of $\mathbf{t}(\alpha)$ as it does not commute with $\mathbf{t}(\beta)$ and $\mathbf{t}(\delta)$. Among the dyonic excitations, there is a simple one that is just a superposition of β^* and δ^* dyons. Once the magnitudes of their electric charges are not identical, their relative charge is nonzero. This corresponds to the excitation due to the $t^3(\alpha)$ rotation. Clearly, this configuration would preserve the axial symmetry. In Sec. V the motion that changes D and this relative charge will be described by a flat two-dimensional surface of revolution. Especially the configuration with relative electric charge is spherically symmetric when $D=0$, which is consistent with the fact that the spherically symmetric solution is not invariant under the global gauge rotations.

On the other hand, our trigonometric case (35) is more complicated. Equation (31) at $(z,0,0)$ becomes

$$\dot{w}_1 - z w_1 + \frac{1}{2} D \tan(Dt) w_1 = 0, \tag{37}$$

$$\dot{w}_2 - z w_2 - \frac{1}{2} D \tan(Dt) w_2 - D \sec(Dt) w_3 = 0, \tag{38}$$

$$\dot{w}_3 + z w_3 - \frac{1}{2} D \tan(Dt) w_3 - D \sec(Dt) w_2 = 0, \tag{39}$$

$$\dot{w}_4 + z w_4 + \frac{1}{2} D \tan(Dt) w_4 = 0. \tag{40}$$

Notice that Eqs. (37) and (40) are not coupled with anything else, while Eqs. (38) and (39) are only coupled among themselves. Thus, after an SU(4) gauge transformation the Higgs field has the form

$$\Phi = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & * \end{pmatrix}, \tag{41}$$

where an asterisk indicates a nonvanishing entry. Since $\Phi^T J + J\Phi = 0$ with J in Eq. (8), we get $\Phi_{33} = -\Phi_{22}$ and $\Phi_{44} = -\Phi_{11}$. From Eq. (37) we can easily obtain

$$\Phi_{22} = -\frac{f(z) - f(-z)}{g(z) + g(-z)}, \tag{42}$$

where

$$f(z) = e^{2z}\{[(2z+1)D^2 + 4z^2(2z-1)]\cos D + D[D^2 + 4z(z-1)]\sin D\}, \tag{43}$$

$$g(z) = e^{2z}(D^2 + 4z^2)(2z \cos D + D \sin D). \tag{44}$$

We are not going to pursue the details for the corner 2×2 matrix part of Φ, which describes the non-Abelian part. Like in the case of Ref. [12], we believe that the trigonometric data correspond to the situation when the energy density is maximized on a ring around the axis of symmetry, even though we have not done the numerical computation to check this. When $D=0$, the configuration is spherically symmetric. When $D \to \pi/2$, we will see in a moment that our result approaches the Atiyah-Hitchin case. That case, when axially symmetric, has a ringlike energy distribution. Thus symmetry and continuity imply the ringlike energy distribution for the trigonometric case.

At the limit $D \to \pi/2$, Eq. (42) becomes

$$\Phi_{22} = -\left[\tanh(2z) - \frac{z}{z^2 + \left(\frac{\pi}{4}\right)^2} \right], \tag{45}$$

which is exactly the result of two SU(2) monopoles [24]. Meanwhile Eqs. (38) and (39) lead to $\Phi_{11} = -\Phi_{44} = -1$ and $\Phi_{14} = \Phi_{41} = 0$ at $D = \pi/2$. Thus the Higgs field (41) along the symmetric axis becomes the Higgs field for charge two-SU(2)-monopole configuration.

As a general verification of the suggestion made above, let us check whether the three-monopole case degenerates into the SU(2) result when $k=0$, $D=\pi/2$, or more generally $D \to K(k)$. In this limit, Nahm data (21) approach

$$f_1, f_2, f_3 \approx -\frac{1}{1+t} \tag{46}$$

near $t=-1$ and

$$-f_1, f_2, f_3 \approx -\frac{1}{1-t} \tag{47}$$

near $t=1$. These are exactly the boundary conditions satisfied by Nahm data for two identical monopoles in the SU(2) case [2,16].

This is a good place to introduce a geometric picture of the non-Abelian cloud. The boundary value (11) is identified with the position (27) of the massless monopole in the center-of-mass frame. The position $\boldsymbol{\zeta}$ of the massless monopole changes under the gauge transformation (22). For an SU(2) transformation $g(t)$ such that $g^{-1}(1) e_i g(1) = \mathcal{R}_{ij} e_j$, $\zeta_i = f_i \mathcal{R}_{i2}$. Thus the SU(2) orbit of the position (27) would be an ellipsoid defined by

$$\frac{\zeta_1^2}{f_1(1)^2} + \frac{\zeta_2^2}{f_2(1)^2} + \frac{\zeta_3^2}{f_3(1)^2} = 1. \tag{48}$$

The size of this ellipsoid would indicate the size of the non-Abelian cloud [7]. [The ellipsoid for Dancer's case is similarly given with $f_i(3)$ replacing $f_i(1)$. This ellipsoid for the spherically symmetric solution has nonzero size.]

For the spherically symmetric solution with $D=0$, this ellipsoid collapses to a point at origin, indicating that the massless monopole is at the origin. Indeed, it is consistent with the picture that all magnetic charges lie at the origin for this solution. For the hyperbolic solution with $k=1$, this ellipsoid collapses into a line connecting two β^* monopoles. Especially when the α^* monopole lies at the end, it is a superposition of β^* and δ^* monopoles. For the trigonometric case with $k=0$, the ellipsoid becomes axially symmetric around the x_1 direction. In the Atiyah-Hitchin limit $D \to K(k)$, the size of this ellipsoid becomes infinite, implying that the massless monopole has been sent to spatial infinity.

From this analysis of various limits, a fairly consistent meaning of two parameters k and D emerges. When we examine the moduli space metric in the next section, we will see that a somewhat richer picture emerges. Figure 2 shows the k-D space. The spherically symmetric case corresponds to the line $D=0$. The trigonometric case lies on the line $k=0$ and $0<D<\pi/2$ and the hyperbolic case lies on the line $k=1$. The Atiyah-Hitchin case corresponds to the curve $D=K(k)$.

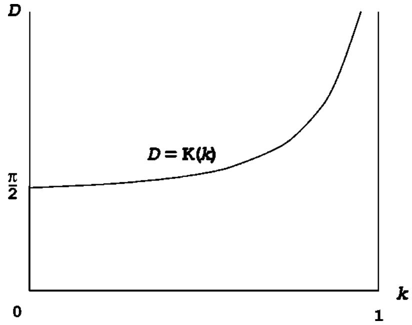

FIG. 2. k-D space.

V. MODULI SPACE METRIC

Now let us turn our attention to the metric of the moduli space. By using centered Nahm data, we work in the center-of-mass frame of monopoles. The relative moduli space M^8 of Nahm data should isometrically correspond to the relative moduli space of the monopole dynamics. The metric for the center-of-mass motion is flat and we expect that

$$M^{12}=R^3\times\frac{S^1\times M^8}{\Delta}, \tag{49}$$

where Δ is a discrete subgroup, about which we are not concerned here. Our work of finding the moduli space metric is greatly facilitated by the works done by Dancer [12] and Irwin [25]. Their general derivation works equally well with our problem. However, our detailed results are different from theirs. For the sake of completeness, we present their derivation more explicitly as the way applied to our case.

To calculate the metric of the relative moduli space M^8, let us define tangent vectors of M^8. A tangent vector $\mathbf{Y}=(Y_1,Y_2,Y_3,Y_4)$ must satisfy the linearized Nahm's equations

$$\dot{Y}_i+[Y_4,T_i]+[T_4,Y_i]=\epsilon_{ijk}[T_j,Y_k]. \tag{50}$$

Since the moduli space M^8 is defined by gauge-equivalent Nahm data, the tangent vector should be orthogonal to infinitesimal gauge transformations δT_μ in $\mathcal{G}_0$, that is,

$$\sum_{\mu=1}^{4}\langle Y_\mu,\delta T_\mu\rangle=0, \tag{51}$$

where the orthogonality is defined with the flat metric on the infinite-dimensional affine space [18,20]

$$ds^2(\mathbf{Y},\mathbf{Y}')=\sum_\mu\langle Y_\mu,Y'_\mu\rangle. \tag{52}$$

Thus Eq. (51) takes an explicit form

$$\dot{Y}_4+\sum_{\mu=1}^{4}[T_\mu,Y_\mu]=0. \tag{53}$$

The procedure of solving Eqs. (50) and (53) for tangent vectors has been described in Ref. [12]. In general, Y_μ can be expressed as $Y_\mu=y_{\mu j}(e_j/2)$. Substituting this expression into Eqs. (50) and (53), we get four linear differential equations, whose nonsingular solutions for Nahm data (21), parametrized by eight real parameters m_μ,n_μ, are

$$Y_1=\frac{1}{2}\left[\dot{f}_1I_1e_1+\left(\dot{f}_2I_2+\frac{m_2}{f_2}\right)e_2+\left(\dot{f}_3I_3+\frac{n_3}{f_3}\right)e_3\right],$$

$$Y_2=\frac{1}{2}\left[-\dot{f}_1I_2e_1+\left(\dot{f}_2I_1+\frac{m_1}{f_2}\right)e_2-\left(\dot{f}_3I_4+\frac{n_4}{f_3}\right)e_3\right],$$

$$Y_3=\frac{1}{2}\left[-\dot{f}_1I_3e_1+\left(\dot{f}_2I_4+\frac{m_4}{f_2}\right)e_2+\left(\dot{f}_3I_1+\frac{n_1}{f_3}\right)e_3\right],$$

$$Y_4=\frac{1}{2}\left[\dot{f}_1I_4e_1+\left(\dot{f}_2I_3+\frac{m_3}{f_2}\right)e_2-\left(\dot{f}_3I_2+\frac{n_2}{f_3}\right)e_3\right], \tag{54}$$

where

$$I_\mu(t)=\int_0^t dt'\left(\frac{m_\mu}{f_2(t')^2}+\frac{n_\mu}{f_3(t')^2}\right). \tag{55}$$

The lower bound of $I_\mu(t)$ is chosen so that they are odd functions. This makes Y_μ satisfy the compatibility condition $Y_\mu(-t)^T=CY_\mu(t)C^{-1}$, which is implied from Eq. (10).

The metric on the moduli space M^8 is induced from the flat metric (52) on the infinite-dimensional affine algebra. With our solutions (54), the general result is

$$ds^2(\mathbf{Y},\mathbf{Y}')=\sum_{\mu=1}^{4}[(g_1+g_1^2X)m_\mu m'_\mu+(g_2+g_2^2X)n_\mu n'_\mu$$
$$+g_1g_2X(m_\mu n'_\mu+n_\mu m'_\mu)], \tag{56}$$

where

$$X(k,D)=f_1(1)f_2(1)f_3(1),$$

$$g_1(k,D)=\int_0^1\frac{dt}{f_2(t)^2},$$

$$g_2(k,D)=\int_0^1\frac{dt}{f_3(t)^2}. \tag{57}$$

We can calculate the metric by finding the tangent vector at a generic point of M^8, which can be obtained by the $SO(3)\times SO(3)$ spatial and gauge rotations of Nahm data (21). Due to the $SO(3)\times SO(3)$ symmetry of the metric, the general metric can be found if it is known near the identity. We want to relate the coordinates m_μ,n_μ of the tangent space at the specific point to the infinitesimal changes of the parameters k,D and the infinitesimal $SO(3)\times SO(3)$ transformations [25]. This corresponds basically the rotation of a rigid body around three principal axes. Similar to the rigid-body case, we can find the metric once we know the moment of inertia around each principal axis, which are the coordinate axes for our Nahm data (15) and (21) [12]. The kinetic part for the rigid-body case is expressed in terms of the left invariant one-forms

$$\sigma_1 = -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\varphi,$$

$$\sigma_2 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\varphi,$$

$$\sigma_3 = d\psi + \cos\theta\, d\varphi, \tag{58}$$

which correspond to the infinitesimal spatial rotations around three principal axes. The corresponding left-invariant one-forms for the gauge rotations are denoted as $\check{\sigma}_i$. The relations we seek are

$$m_1 = -\frac{1}{2} d\lambda_1,$$

$$n_1 = -\frac{1}{2} d\lambda_2,$$

$$m_2 = \lambda_1 \sigma_3,$$

$$n_2 = -\frac{g_1\lambda_1}{1+g_2X}\sigma_3 + \frac{1}{\sqrt{g_2+g_2^2X}}\left\{ b_3\sigma_3 - c_3\left(\frac{f_1(1)}{f_2(1)}\sigma_3 - \check{\sigma}_3\right)\right\},$$

$$m_3 = \frac{g_2\lambda_2}{1+g_1X}\sigma_2 - \frac{1}{\sqrt{g_1+g_1^2X}}\left\{ b_2\sigma_2 - c_2\left(\frac{f_1(1)}{f_3(1)}\sigma_2 - \check{\sigma}_2\right)\right\},$$

$$n_3 = -\lambda_2\sigma_2,$$

$$m_4 = -\frac{g_2(\lambda_1-\lambda_2)}{g_1+g_2}\sigma_1 - \frac{1}{\sqrt{(g_1+g_2)(1+(g_1+g_2)X)}} \times\left\{ b_1\sigma_1 - c_1\left(\frac{f_3(1)}{f_2(1)}\sigma_1 - \check{\sigma}_1\right)\right\},$$

$$n_4 = \frac{g_1(\lambda_1-\lambda_2)}{g_1+g_2}\sigma_1 - \frac{1}{\sqrt{(g_1+g_2)(1+(g_1+g_2)X)}} \times\left\{ b_1\sigma_1 - c_1\left(\frac{f_3(1)}{f_2(2)}\sigma_1 - \check{\sigma}_1\right)\right\}, \tag{59}$$

where λ_1, λ_2 are given in Eq. (25),

$$b_1 = k^2D^2\sqrt{\frac{g_1^2}{(g_1+g_2)(1+(g_1+g_2)X)}},$$

$$b_2 = \frac{g_2D^2}{\sqrt{g_1+g_1^2X}},$$

$$b_3 = \frac{g_1(1-k^2)D^2}{\sqrt{g_2+g_2^2X}} \tag{60}$$

and

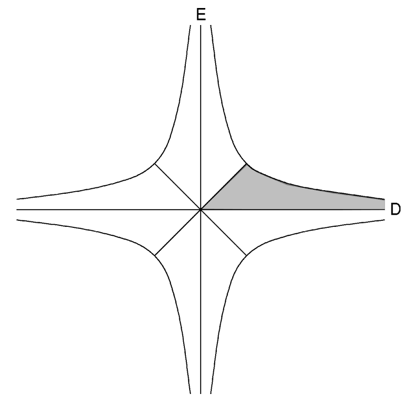

FIG. 3. Sketch of the geodesically complete space in D-E coordinates. The shaded region corresponds to the $N^5/\mathrm{SO}(3)$ space.

$$c_1 = f_2(1)f_3(1)\sqrt{\frac{g_1+g_2}{1+(g_1+g_2)X}},$$

$$c_2 = \frac{\sqrt{g_1+g_1^2X}}{f_2(1)g_1},$$

$$c_3 = \frac{\sqrt{g_2+g_2^2X}}{f_3(1)g_2}. \tag{61}$$

Once we replaced the parametrization (59) into the metric (56), we would have obtained the explicit form for the metric of the moduli space M^8. Rather, let us study the metric step by step. The two-dimensional space $\mathcal{Y}^2$ is the geodesically complete space made of eight copies of the k-D space, $N^5/\mathrm{Sp}(3)$, as discussed after Eq. (21). This space describes the motion of the monopoles with the vanishing SU(2) electric charge and zero angular momentum. The metric of this space obtained from Eqs. (25), (56), and (59) is

$$ds^2_{\mathcal{Y}^2} = \frac{1}{4}\{X(g_1d\lambda_1 + g_2d\lambda_2)^2 + g_1d\lambda_1^2 + g_2d\lambda_2^2\}. \tag{62}$$

Figure 3 shows this space in terms of two coordinates, which in the shaded region are D and $E \equiv \sqrt{1-k^2}D$. The above metric at origin is smooth with D,E playing Cartesian coordinates. The origin corresponds to the spherically symmetric configuration. Two coordinate axes correspond to two hyperbolic configurations, which are symmetric along real spatial x_2, x_3 coordinate axes. The diagonal lines correspond to one trigonometric one, which is symmetric along the real spatial x_1 axis. The boundary curves correspond to the Atiyah-Hitchin configurations, where the massless monopole has been moved to spatial infinity.

While we have not studied in detail the geodesic motion on this space, one can see from symmetry that the trigonometric solutions with velocity pointing to the origin will remain trigonometric after the configuration passes through the origin. With a similar velocity, the hyperbolic solutions will remain hyperbolic, which is consistent with the view that two monopoles without gauge charge do not interact for this case. This is in contrast with Dancer's case where the trigonometric configuration changes to the hyperbolic one and vice versa. When the cloud size becomes large, the configu-

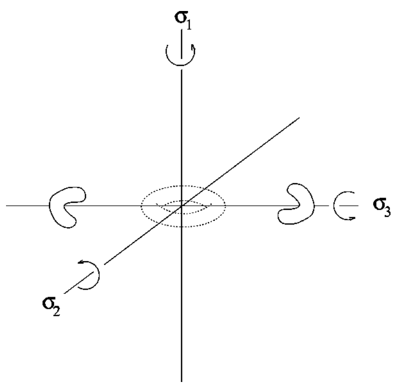

FIG. 4. Massive monopole with a finite-size cloud. The central doughnut indicates the symmetric axis x_1.

ration would approach the Atiyah-Hitchin configuration and the boundary curve shows the 90° scattering of these monopoles.

The metric on $N^5 = M^8/\mathrm{SU}(2)$ with the $Z_2 \times Z_2$ isotropic group is

$$ds^2_{N^5} = ds^2_{\mathcal{Y}^2} + a_1\sigma_1^2 + a_2\sigma_2^2 + a_3\sigma_3^2, \tag{63}$$

with

$$a_1 = k^4 D^4 \frac{g_1 g_2}{g_1 + g_2},$$

$$a_2 = D^4 \left\{ g_2 + \frac{g_2^2 X}{1 + g_1 X} \right\},$$

$$a_3 = (1-k^2)^2 D^4 \left\{ g_1 + \frac{g_1^2 X}{1 + g_2 X} \right\}. \tag{64}$$

Here one uses the orthogonality condition for the tangential vectors of N^5 to that of gauge rotation [12,25], which can be found from Eq. (59) by dropping terms depending on b_i and c_i. There is no cross term for the invariant one-forms, which is consistent with $Z_2 \times Z_2$ isotropy group of N^5. This metric describes the monopole dynamics with zero SU(2) gauge charge, perhaps with nonzero orbital angular momentum. Figure 4 shows two massive monopoles (two half doughnuts on the x_3 axis) with generic cloud size and three principal axes. In zero cloud size $k=1$, the metric is symmetric under the rotation around the x_3 axis so that $a_1 = a_2$ and $a_3 = 0$. In the trigonometric case $k=0$, the metric is symmetric under the rotation around the x_1 axis so that $a_1 = 0$ and $a_2 = a_3$.

The distance between two massive monopoles is defined up to the monopole core size. We guess this distance as $r \approx \sqrt{a_1} \approx \sqrt{a_2}$ because the moment of inertia for a point particle would be mass times the square of distance from the origin. We will see later that the above approximation is true when the non-Abelian cloud size is very small or very large.

From Eqs. (56) and (59) we get the full metric on M^8, which is

$$\begin{aligned} ds^2_{M^8} = {} & \frac{1}{4}\{X(g_1 d\lambda_1 + g_2 d\lambda_2)^2 + g_1 d\lambda_1^2 + g_2 d\lambda_2^2\} + a_1\sigma_1^2 \\ & + a_2\sigma_2^2 + a_3\sigma_3^2 + \left\{ b_1\sigma_1 - c_1\left(\frac{f_3(1)}{f_2(1)}\sigma_1 - \check{\sigma}_1\right)\right\}^2 \\ & + \left\{ b_2\sigma_2 - c_2\left(\frac{f_1(1)}{f_3(1)}\sigma_2 - \check{\sigma}_2\right)\right\}^2 \\ & + \left\{ b_3\sigma_3 - c_3\left(\frac{f_1(1)}{f_2(1)}\sigma_3 - \check{\sigma}_3\right)\right\}^2. \end{aligned} \tag{65}$$

This metric is hyperkähler. The isometric group is $\mathrm{SO}(3) \times \mathrm{SO}(3)$. The SO(3) global gauge transformations are triholomorphic and the SO(3) spatial rotations rotate three complex structures of the manifold. There are several interesting limits of this metric. When the cloud size is smallest with $k=1$, its Nahm data are the hyperbolic case (34) and the above metric becomes

$$\begin{aligned} ds^2_{\mathrm{hyper}} = {} & dD^2 + D^2\sigma_1^2 + D^2\sigma_2^2 + D\tanh D\ \check{\sigma}_1^2 + D\coth D\ \check{\sigma}_2^2 \\ & + \check{\sigma}_3^2. \end{aligned} \tag{66}$$

The moments of inertia for internal gauge transformations are nonzero exactly as we argued in Sec. IV. Especially for spherically symmetric case with $D=0$, the coefficient of $\check{\sigma}_1$ vanishes, but those of $\check{\sigma}_2$ and $\check{\sigma}_3$ are nonvanishing and become identical, implying the S^2 gauge orbit space. In large separation $D \gg 1$, the inertia for $\check{\sigma}_1$ and $\check{\sigma}_2$ become identical. The inertia for $\check{\sigma}_3$ is constant, which corresponds to $t^3(\alpha)$ dyonic excitations discussed in the paragraph after Eq. (36). We also see that $\sqrt{a_2} = \sqrt{a_2} = D$ is indeed the distance between two massive monopoles.

When the distance between two β^* monopoles is smallest, its Nahm data are the trigonometric case (35). In this case the metric (66) takes a rather complicated form. While the explicit formula can be obtained easily, we will not bother to write it down here. We simply note that the moments of inertia along x_2 and x_3 axes are identical because the trigonometric configurations are symmetric under rotation around the x_1 axis.

There are two types of surface of evolution, corresponding to the hyperbolic and trigonometric cases. When we include internal global gauge rotations that preserve the axial symmetry, we obtain two-dimensional surfaces of revolution. The two-dimensional metric for the trigonometric case with $k=0$ is

$$\begin{aligned} ds^2_{\mathrm{trig2}} = {} & \sec^2 D\ \ (1 + D\tan D)\left(1 + \frac{\sin 2D}{2D}\right) \\ & \times\left[dD^2 + \frac{D^2(\sigma_1 - \check{\sigma}_1)^2}{(1 + D\tan D)^2}\right], \end{aligned} \tag{67}$$

where $\sigma_1 - \check{\sigma}_1$ can be put into a rotation $d\alpha$. As $D \to \pi/2$, the metric (67) becomes

$$ds^2 = d\rho^2 + \frac{1}{4}\rho^2 d\alpha^2, \tag{68}$$

where $\rho=2\sqrt{\pi/(\pi-2D)}$. In this limit the massless monopole moves out from localized massive monopoles and so the non-Abelian cloud is expected to become more spherical with the flat R^4 moduli space as in Ref. [7]. The above metric is then a section of R^4 with a radial variable ρ as we will see in a moment. In the physical space, the massless cloud size is of order ρ^2. The non-Abelian component of the gauge field will change its behavior from $1/r$ to $1/r^2$ as one crosses this radius ρ^2. Another axially symmetric case is hyperbolic one with $k=1, 0<D<\infty$, whose metric is

$$ds^2_{\rm hyper2}=dD^2+\check{\sigma}_3^2. \tag{69}$$

Clearly this flat metric is a part of the metric (66).

The limit of large cloud size can be found in the region where $K(k)-D \ll 1$. In Sec. IV, we argued that Nahm data in this case approach those for the Atiyah-Hitchin case. In this limit one can show easily that the metric (65) becomes

$$ds^2=d\rho^2+\frac{\rho^2}{4}\{(\sigma_1-\check{\sigma}_1)^2+(\sigma_2+\check{\sigma}_2)^2+(\sigma_3+\check{\sigma}_3)^2\}$$
$$+\frac{b^2}{K^2}dK^2+a^2\sigma_1^2+b^2\sigma_2^2+c^2\sigma_3^2+O(\rho^{-1}), \tag{70}$$

where $\rho=2\sqrt{D/(K-D)}$ and

$$a^2=\frac{K(K-E)[E-(1-k^2)K]}{E}, \tag{71}$$

$$b^2=\frac{EK(K-E)}{E-(1-k^2)K}, \tag{72}$$

$$c^2=\frac{EK[E-(1-k^2)K]}{K-E}, \tag{73}$$

with the second complete elliptic integral $E=\int_0^{\pi/2}d\theta\sqrt{1-k^2\sin^2\theta}$. This shows that the asymptotic space is a direct product of R^4 and the Atiyah-Hitchin space. As in Ref. [7], we expect that the metric of the massless cloud space approaches that of flat R^4, which is exactly what the above limit shows. A combination of orbital and gauge angular variables needs to be introduced [25] to make this R^4 explicit. We note that $\sqrt{a_1}=a$ and $\sqrt{a_2}=b$. The distance between two massive monopoles is given approximately as $r\approx a\approx b\approx -\ln\sqrt{1-k^2}$ [20].

The part of the modulus space metric we can calculate independently from Nahm's formalism is the asymptotic metric, which is valid when the mutual distances between monopoles are large. This can be done by studying the interaction between dyons in large separation [26,11] and taking the massless limit. In the center-of-mass frame, the relative positions between the massive β^* monopoles and the massless α^* monopole are $\mathbf{r}_1$ and $\mathbf{r}_2$ as shown in Fig. 5. The relative position between two massive monopoles is $\mathbf{r}=\mathbf{r}_1+\mathbf{r}_2$.

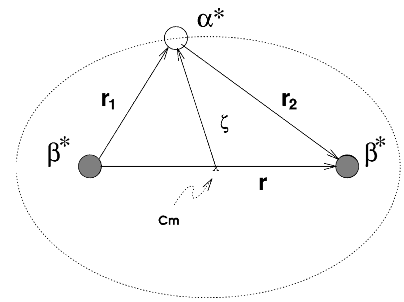

FIG. 5. Parameters of the asymptotic metric. The center of mass is at the middle of the line connecting two massive β^* monopoles.

In terms of the relative positions and the relative angles $\psi_a, a=1,2$, the asymptotic form of the metric for the relative moduli space M^8 is

$$ds^2=\sum_{a,b}^{2}[G_{ab}d\mathbf{r}_a\cdot d\mathbf{r}_b+(G^{-1})_{ab}D\psi_a D\psi_b], \tag{74}$$

where

$$G_{ab}=\begin{pmatrix} \frac{1+2\epsilon}{1+\epsilon}+\frac{1}{r_1}-\frac{1}{r} & \frac{1}{1+\epsilon}-\frac{1}{r} \\ \frac{1}{1+\epsilon}-\frac{1}{r} & \frac{1+2\epsilon}{1+\epsilon}+\frac{1}{r_2}-\frac{1}{r}\end{pmatrix}, \tag{75}$$

$$D\psi_a=d\psi_a+\mathbf{w}(\mathbf{r}_a)\cdot d\mathbf{r}_a-\mathbf{w}(\mathbf{r})\cdot d\mathbf{r}, \tag{76}$$

with the Dirac potential $\mathbf{w}$ such that $\nabla\times\mathbf{w}(\mathbf{r})=\nabla(1/r)$. Here, for the sake of later use, we have introduced a parameter ϵ that is propotional to the ratio of the α^* monopole mass and β^* monopole mass. The limit where α^* monopoles become massless is then $\epsilon\rightarrow 0+$. If we have removed the direct interaction between two identical massive monopoles, the above metric is identical to the Taubian-Calabi metric of the $SU(4)\rightarrow U(1)\times SU(2)\times U(1)$ case. Since the non-Abelian cloud of a massless monopole is independent of the direct interaction, the SU(2) orbit on the non-Abelian cloud would again be the three-dimensional ellipsoid defined by $r_1+r_2=$const, as shown in Fig. 5. This ellipsoid would be the limit of the ellipsoid (48), where it becomes symmetric around the x_3 axis. This fact can be seen easily by adapting the argument for the SU(4) case in Ref. [7]. In the large cloud size limit, one can compare the exact metric (65) and the above one. We see $K/(K-D)\approx r_1+r_2$ and $r\approx K(k)\approx -\ln\sqrt{1-k^2}$. The condition $r_1+r_2\gg r$ for the large cloud size becomes $K(k)-D\ll 1$.

Now we are in position to learn more about the four-dimensional space $M^4(\boldsymbol{\zeta})$ defined by the moment map (26) generated from the τ_3 U(1). For Nahm data (15) and (21) we get $\boldsymbol{\zeta}=(0,\zeta,0)$ with

$$\zeta=D\sqrt{1-k^2}\frac{1}{{\rm cn}_k(D)}. \tag{77}$$

The general Nahm data are obtained from that in Eq. (21) by spatial and gauge rotations. Thus $\boldsymbol{\zeta}$ would be a function of rotational and gauge parameters. With our choice of U(1) and the Nahm data (21), it is easy to find the quotient space $M^4(\boldsymbol{\zeta})=\mu^{-1}(\boldsymbol{\zeta})/\mathrm{U}(1)$ at $\boldsymbol{\zeta}=\mathbf{0}$, which corresponds to $k=1$ from Eq. (77). This corresponds to the hyperbolic data (34) with the minimal size of non-Abelian cloud. The moduli space metric is that (66) for the hyperbolic data. Among them, the gauge rotation by τ_1 changes $\boldsymbol{\zeta}$ away from zero and so the corresponding $\check{\sigma}_1$ part should be dropped. Dividing by the U(1) group of τ_3 implies dropping the $\check{\sigma}_2$ term from the metric. The resulting four-dimensional space is the flat $R^3\times S^1$ with the metric

$$ds^2=dD^2+D^2(\sigma_1^2+\sigma_2^2)+\check{\sigma}_3^2. \tag{78}$$

On the other hand, when $\zeta=\infty$, we have $D=K(k)$, which means that the massless monopole has been removed, resulting in the Atiyah-Hitchin metric. Thus we see that $M^4(\boldsymbol{\zeta})$ interpolates between $M^4(0)=R^3\times S^1$ and the Atiyah-Hitchin space $M^4(\infty)$.

We find another argument for the above result (78) by considering the asymptotic form of the metric (74). Among two conserved U(1) generated by $\psi_a\rightarrow\psi_a+\delta_a$, the relative electric charge of two massive monopoles, which generates $\psi_a\rightarrow\psi_a+\delta$, is not conserved when the short-distance correction is included. However, the difference between total charge of two β^* monopoles and that of an α^* monopole, which generates the transformation $\psi_1\rightarrow\psi_1+\delta$ and $\psi_2\rightarrow\psi_2-\delta$, will remain exact even when a short distance correction is included. The moment map of this U(1) symmetry, which can be obtained easily from the hyperkähler one-forms [26,9], is

$$\boldsymbol{\zeta}=\frac{\mathbf{r}_1-\mathbf{r}_2}{2}. \tag{79}$$

This is the position of the massless monopole as shown in Fig. 5. (Even in the maximally broken case, the above moment map is correct.) As $\boldsymbol{\zeta}$ increases from zero to infinity, the size of the non-Abelian cloud increases from zero to infinity, consistent with the picture discussed in the preceding paragraph. Also, we can trivially obtain the asymptotic form of the metric for $M^4(\zeta)$,

$$ds^2=G\; d\mathbf{r}^2+G^{-1}(d\psi+\mathbf{W}\cdot d\mathbf{r})^2, \tag{80}$$

where

$$G=1+\frac{1}{2|\mathbf{r}+2\boldsymbol{\zeta}|}+\frac{1}{2|\mathbf{r}-2\boldsymbol{\zeta}|}-\frac{1}{|\mathbf{r}|} \tag{81}$$

and $\mathbf{W}$ is decided from the relation $\nabla G=\nabla\times\mathbf{W}$. Clearly, this asympototic form is hyperkähler, as it satisfies the condition for the toric hyperkähler space [26]. This metric is correct whether or not the α^* monopole is massless. In massless case, this hyperkähler quotient can be obtained by holding the position of the massless monopole at $\boldsymbol{\zeta}$ relative to the center of mass and let massive monopoles move around, interacting with each other and with the massless monopole. In massive case, we are holding the $\boldsymbol{\zeta}$. The position of the center of mass would lie between the position of the α^* monopole and the middle of line connecting two β^* monopoles.[2] When $\boldsymbol{\zeta}=\mathbf{0}$, the metric (80) becomes the flat metric (78).

VI. CONCLUSION

We have studied a purely Abelian BPS monopole configuration made of two identical massive monopoles and one massless monopole in the theory where the gauge group Sp(4) is spontaneously broken to SU(2)×U(1) along a short root. We approached this problem by finding the solutions for the corresponding Nahm equations under proper boundary and compatibility conditions. We have used the ADHMN construction to get the spherically and axially symmetric field configurations, which are consistent with the field theory picture. From the analysis of the axially symmetric solutions, we have come to understand the role of the non-Abelian cloud and its size. Then the explicit form of the metric on the eight-dimensional moduli space of relative motion is found. By studying the metric in various limits, we see that this metric for the moduli space of the Nahm data is shown to be consistent with what is expected from the monopole dynamics.

We have also studied the metric of various submanifolds of this space. Our work provides further support to the idea that Nahm's approach for the BPS monopole configurations and their modulus spaces is valid in general. Our work leads also to some insight into the characteristics of the non-Abelian cloud and the gauge orbit. It is interesting to note that the spherically symmetric solution has nonzero inertia for some unbroken gauge transformations.

From the previous experiences we now see how in principle one may find the moduli space metric of two identical massive and one massless monopoles in the theory with $G_2\rightarrow\mathrm{SU}(2)\times\mathrm{U}(1)$. To get this one may start from the theory with $\mathrm{SO}(8)\rightarrow\mathrm{SU}(2)^3\times\mathrm{U}(1)$ with two identical massive and three distinct massless monopoles. If one identifies two massless monopoles, then the configuration would be that of two massive and two distinct massless monopoles in the theory with $\mathrm{SO}(7)\rightarrow\mathrm{SU}(2)^2\times\mathrm{U}(1)$. After further identification of all massless monopoles, one would get the desired configuration in the theory with G_2.

The hyperkähler quotient of the eight-dimensional relative moduli spaces of these configurations is a four-dimensional hyperkähler space. To find the $M^4(\boldsymbol{\zeta}=0)$ one can consider the asymptotic form of the metric for two massive monopoles with a minimum size cloud. (We overlap the massless monopole on one of the massive monopoles.) They are $R^3\times S^1$, Taub–Newman-Unti-Tamburine (NUT) or a double covering of Atiyah and Hitchin, depending on whether they are associated with the gauge group Sp(4), G_2, or SU(3), respectively. When the cloud size becomes large, all these

[2] A somewhat different approach has been taken in Ref. [22]. There the quotient space for Dancer's case is obtained by taking the infinite mass limit of the corresponding α^* monopole. This fixes the absolute position of the α^* monopole rather than $\boldsymbol{\zeta}$, which is its position relative to the center of mass of two massive β^* monopoles. The resulting quotient metric seems not to be hyperkähler, as it does not conform to the generic metric for toric hyperkähler spaces.

four-dimensional spaces approach the Atiyah-Hitchin space.

Another direction to explore is to find the moduli space in the case when α^* monopoles become massive so that there are two identical massive monopoles and one distinct massive monopole. We think that the moduli space in the theory where $\mathrm{Sp}(4)\rightarrow\mathrm{U}(1)^2$ is simpler than the similar problem in the theory with $\mathrm{SU}(3)\rightarrow\mathrm{U}(1)^2$. Also this moduli space has a role to play in the $N=2$ S duality [27]. Finding the moduli space will be a challenge. Finally, it would be very interesting to find some structure of the moduli space of three massive and three massless monopoles in the theory where $\mathrm{SU}(4)\rightarrow\mathrm{SU}(3)\times\mathrm{U}(1)$. We know the asymptotic form of the metric and it may be good enough. As argued in the Introduction, these configurations can be regarded as a magnetic dual of baryons and might lead to further insight on the structure of baryons.

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