Finite Temperature Chern-Simons Coefficient

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We compute the exact finite temperature effective action in a $(0 + 1)$-dimensional field theory containing a topological Chern-Simons term, which has many features in common with $(2 + 1)$-dimensional Chern-Simons theories. This exact result explains the origin and meaning of puzzling temperature dependent coefficients found in various naive perturbative computations in the higher dimensional models. [S0031-9007(97)03131-1]

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There are many examples in physics in which the classical Lagrange density contains a term that is not strictly invariant under a certain transformation (for example, a "large" gauge transformation), but the classical action changes by a constant that takes discrete values associated with the "winding number" of the transformation. For such a system the quantum theory is formally invariant provided the amplitude $\exp[i(\text{action})]$ is invariant; thus, invariance of the quantum theory can be maintained provided the coefficient of the noninvariant term in the Lagrange density is chosen to take appropriate discrete values. This argument is familiar in the theory of the Dirac magnetic monopole, and in Chern-Simons theories [1]. It is important to ask what happens to this discretization condition when quantum interactions are taken into account. For example, the quantum effective action may contain induced terms of the same noninvariant form, but with a new coefficient. This subject of induced topological terms is relatively well understood in various examples of zero temperature quantum field theory [2–4]. However, there is currently a great deal of confusion in the corresponding theories at nonzero temperature. Typically [5–8], a naive perturbative computation that mimics the zero temperature computation leads to an induced topological term equal to the zero temperature induced topological term, but multiplied by an extra factor of $\tanh(\beta|m|/2)$. Here $\beta = 1/T$ is the inverse temperature, and m is a relevant mass scale. Clearly, this coefficient cannot take only discrete values for all T, even though formal arguments suggest that it should. This dilemma has recently been emphasized [6,9] for the particular case of $(2 + 1)$-dimensional fermion and/or Chern-Simons theories, for which quantum effects may lead to induced Chern-Simons terms. (Related features also appear in monopole and Aharonov-Bohm systems [10–12].) There is one opinion that anyonic superfluidity should break down at any finite temperature due to this anomaly [13]. There is an opposite opinion that there is no such temperature dependent anomaly due to some "nonperturbative" physics. However, we feel that the discussion thus far has missed an essential point. To illustrate this, we consider a simple analog of the Chern-Simons system, which has the advantage that it may be solved exactly and yet it still retains the essential topological complexities of the problem.

Consider a $(0 + 1)$-dimensional field theory of N_f fermions ψ_j, $j = 1, \ldots, N_f$, minimally coupled to a U(1) gauge field A. It is not possible to write a Maxwell-like kinetic term for the gauge field in $0 + 1$ dimensions, but we can write a Chern-Simons term—it is linear in A. (Recall that it is possible to define a Chern-Simons term in odd dimensional space-time. Some features of "Chern-Simons quantum mechanics" have been studied previously [14].) We formulate the theory in Euclidean space (i.e., imaginary time) so that we can go smoothly between nonzero and zero temperature using the imaginary time formalism [15]. The Lagrange density is

$$\mathcal{L} = \sum_{j=1}^{N_f} \psi_j^\dagger (\partial_\tau - iA + m)\psi_j - i\kappa A \,. \qquad (1)$$

There are many similarities between this model and the $(2 + 1)$-dimensional model of fermions coupled to a non-Abelian Chern-Simons gauge field. For example, this model supports gauge transformations with a nontrivial winding number. Under the U(1) gauge transformation $\psi \to e^{i\lambda}\psi$, $A \to A + \partial_\tau \lambda$, the Lagrange density changes by a total derivative and the action changes by

$$\Delta S = -i\kappa \int d\tau\, \partial_\tau \lambda = -2\pi i \kappa N \,, \qquad (2)$$

where $N \equiv \frac{1}{2\pi} \int d\tau\, \partial_\tau \lambda$ is the integer-valued winding number of the topologically nontrivial gauge transformation. Thus, choosing κ to be an integer, the Euclidean action changes by an integer multiple of $2\pi i$, so that the quantum path integral is formally invariant—just as in three dimensional non-Abelian Chern-Simons theories [1].

Under naive charge conjugation C: $\psi \to \psi^\dagger$, $A \to -A$, the fermion mass term and the Chern-Simons term are not

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invariant. This situation is similar to the fermion mass term and the Chern-Simons term in three dimensions, which are not invariant under the parity transformation. In that case, introducing an equal number of fermions of opposite sign mass, the fermion mass term can be made invariant under a generalized parity transformation. Similarly, with an equal number of fermion fields of opposite sign mass, one can generalize charge conjugation to make the mass term invariant in our $(0 + 1)$-dimensional model.

There is a global part of the U(1) symmetry, whose conserved charge is

$$Q_F = \frac{1}{2} \sum_j (\psi_j^\dagger \psi_j - \psi_j \psi_j^\dagger) . \tag{3}$$

The fermionic Hamiltonian is

$$H_F = \frac{m}{2} \sum_j (\psi_j^\dagger \psi_j - \psi_j \psi_j^\dagger) = m Q_F . \tag{4}$$

Both Q_F and H_F change sign under charge conjugation. In addition to the U(1) gauge symmetry, there is a global SU(N_f) flavor symmetry, whose conserved charges are

$$R^a = \sum_{i,j} \psi_i^\dagger T_{ij}^a \psi_j , \tag{5}$$

where the T^a are the generators of SU(N_f) in the fundamental representation.

The canonical commutation relations are $\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$, and the ground state $|0\rangle$ is chosen so that the energy is lowest: $\psi_j|0\rangle = 0$ if $m > 0$, and $\psi_j^\dagger|0\rangle = 0$ if $m < 0$. Then the vacuum expectation value of the fermionic charge Q_F at zero temperature is

$$\langle 0|Q_F|0\rangle = -\frac{m}{2|m|} N_f , \tag{6}$$

while at nonzero temperature

$$\langle 0|Q_F|0\rangle_\beta = -\frac{m}{2|m|} N_f \tanh\left(\frac{\beta|m|}{2}\right). \tag{7}$$

Note that the $T = 0$ answer (6) is regained smoothly in the zero T ($\beta \to \infty$) limit.

We now compute the effective action for this theory:

$$\Gamma[A] = \ln\left[\frac{\det(\partial_\tau - iA + m)}{\det(\partial_\tau + m)}\right]^{N_f}. \tag{8}$$

Recalling that the fermion fields at finite temperature are *antiperiodic*, $\psi(0) = -\psi(\beta)$, the eigenvalues of the operator $\partial_\tau - iA + m$ are

$$\Lambda_n = m - i\frac{a}{\beta} + \frac{(2n-1)\pi i}{\beta}, \qquad n = -\infty, \ldots, +\infty , \tag{9}$$

where $a \equiv \int_0^\beta d\tau\, A(\tau)$. To get these, we can make a small gauge transformation so that A takes the constant value a/β. Then the determinants may be computed as usual [16]

$$\frac{\det(\partial_\tau - iA + m)}{\det(\partial_\tau + m)} = \prod_{n=-\infty}^{\infty} \left[\frac{m - i\frac{a}{\beta} + \frac{(2n-1)\pi i}{\beta}}{m + \frac{(2n-1)\pi i}{\beta}}\right] = \frac{\cosh(\frac{\beta m}{2} - i\frac{a}{2})}{\cosh(\frac{\beta m}{2})} . \tag{10}$$

Thus the exact finite temperature effective action is

$$\Gamma[A] = N_f \ln\left[\cos\left(\frac{a}{2}\right) - i \tanh\left(\frac{\beta m}{2}\right)\sin\left(\frac{a}{2}\right)\right]. \tag{11}$$

It is interesting to notice that $\Gamma[A]$ is not an extensive quantity (i.e., it is not an integral of a density) in Euclidean time. Rather, it is a complicated function of the time integral of A.

In the zero temperature limit, the tanh function reduces to $\tanh(\frac{\beta m}{2}) \to \frac{m}{|m|}$, so that

$$\Gamma[A]_{T=0} = -\frac{i}{2}\frac{m}{|m|} N_f \int d\tau\, A(\tau) . \tag{12}$$

The zero temperature effective action *is* an extensive quantity in Euclidean time. Indeed, $\Gamma[A]_{T=0}$ has the same form as the original Chern-Simons term, and we conclude that the Chern-Simons coefficient κ is shifted in the combined, classical plus effective, action:

$$\kappa \to \kappa - \frac{1}{2}\frac{m}{|m|} N_f . \tag{13}$$

The shift $\delta\kappa$ at $T = 0$ is exactly the charge (6) of the $T = 0$ fermion ground state, which should dominate the zero temperature correction. This is quantized in half integer units. (This is the quantum mechanical analog of the global anomaly [2,16].) If the number of fermion flavors is even, the vacuum charge is an integer and there is no global anomaly. This is the same as in $(2 + 1)$-dimensional fermion-Chern-Simons theories.

The finite temperature effective action is more complicated. An expansion of the exact result (11) in powers of the gauge field yields

$$\Gamma[A] = N_f\left[-\frac{i}{2}\tanh\left(\frac{\beta m}{2}\right)a - \frac{1}{8}\mathrm{sech}^2\left(\frac{\beta m}{2}\right)a^2 - \frac{i}{24}\tanh\left(\frac{\beta m}{2}\right)\mathrm{sech}^2\left(\frac{\beta m}{2}\right)a^3 + \ldots\right]. \tag{14}$$

We can compare this exact result with a standard field theoretic perturbative computation:

$$\Gamma[A] = N_f \ln\det(1 - iSA) = -N_f \sum_{p=1}^{\infty} \frac{i^p}{p}\mathrm{tr}(SASA\ldots SA) , \tag{15}$$

where S is Green's function for the free operator $(\partial_\tau + m)$.

It is instructive to consider first the case in which A is constant: $A = a/\beta$. Then

$$\mathrm{tr}(S^p) = \sum_{n=-\infty}^{\infty} \frac{1}{(\frac{(2n-1)\pi i}{\beta} + m)^p} = \frac{\beta}{2}\frac{(-1)^{p-1}}{(p-1)!}\left(\frac{\partial}{\partial m}\right)^{p-1}\tanh\left(\frac{m\beta}{2}\right) \qquad (16)$$

in which case the perturbative expansion (15) yields the expansion (14) of the exact result.

For a general gauge field $A(\tau)$ [recall that $A(\tau)$ is a periodic function of τ in the imaginary time formulation of the finite temperature theory] we can compute all orders in the perturbation series (15) using Green's function

$$S(\tau-\tau') = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{e^{i(2n-1)\pi i/\beta}}{\frac{(2n-1)\pi i}{\beta}+m} = -\frac{1}{2}\left[\sinh(m|\tau-\tau'|) - \tanh\left(\frac{m\beta}{2}\right)\cosh(m|\tau-\tau'|)\right] + \frac{1}{2}\epsilon_\beta(\tau-\tau')\left[\cosh(m|\tau-\tau'|) - \tanh\left(\frac{m\beta}{2}\right)\sinh(m|\tau-\tau'|)\right]. \qquad (17)$$

Here $\epsilon_\beta(\tau)$ is the periodic step function:

$$\epsilon_\beta(\tau) = \begin{cases} +1, & 0 < \tau < \beta, \\ -1, & -\beta < \tau < 0, \end{cases} \qquad (18)$$

with $\epsilon_\beta(n\beta) \equiv 0$. Thus $S(0) = \frac{1}{2}\tanh(\frac{m\beta}{2})$, and $S(\beta) = -\frac{1}{2}\tanh(\frac{m\beta}{2})$; while for $0 < \tau < \beta$, $S(\tau) = \frac{1}{2}[1 + \tanh(\frac{m\beta}{2})]e^{-m\tau}$, and for $-\beta < \tau < 0$, $S(\tau) = -\frac{1}{2}[1 - \tanh(\frac{m\beta}{2})]e^{-m\tau}$. Equipped with these results for Green's function we can now compute the pth order contribution to the perturbative expansion (15) of the effective action:

$$-\frac{i^p}{p}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\cdots\int_0^\beta d\tau_p\, S(\tau_1-\tau_2)A(\tau_2)S(\tau_2-\tau_3)A(\tau_3)\cdots S(\tau_p-\tau_1)A(\tau_1) = \frac{(-i)^p}{2^p p!}\left[\left(\frac{\partial}{\partial(\frac{m\beta}{2})}\right)^{p-1}\tanh\left(\frac{m\beta}{2}\right)\right]\left[\int_0^\beta A(\tau)\,d\tau\right]^p. \qquad (19)$$

We obtain the same expansion as the expansion (14) of the exact answer (11). Once again we see that the full effective action $\Gamma[A]$ is not an extensive quantity in Euclidean time.

We can alternatively derive the finite temperature effective action from the partition function in a grand canonical ensemble. Consider a constant gauge field $A(\tau) = -i\mu$. Then

$$e^{\Gamma[A]} = \frac{\mathrm{tr}\exp[-\beta H_F + \beta\mu Q_F]}{\mathrm{tr}\exp[-\beta H_F]}, \qquad (20)$$

where H_F is the fermionic Hamiltonian (4) and Q_F is the fermionic charge (3). A straightforward calculation yields

$$e^{\Gamma[A]} = \left[\frac{\cosh\frac{\beta(m-\mu)}{2}}{\cosh\frac{\beta m}{2}}\right]^{N_f} \qquad (21)$$

in agreement with (10) and (11).

Our $(0+1)$-dimensional model is special in the sense that we are able to compute *every order* in perturbation theory, and furthermore we are able to *re-sum* the perturbative expansion to obtain the *exact* effective action. In higher dimensional examples it is generally not possible to compute the exact effective action, because of the extra momentum integrations and the additional tensor or spinor structure. Thus, a field theoretic computation of the finite temperature effective action in $2+1$ dimensions [5–9] is a perturbative one that typically looks only for the *lowest order* term, which happens to be the same as the original Chern-Simons term. Applying this same philosophy to the example treated here, we would (erroneously) conclude that the effective action is just the first term in the expansion (14), and hence that the Chern-Simons coefficient is shifted by a temperature-dependent amount

$$\kappa \to \kappa - \frac{1}{2}\frac{m}{|m|}\tanh\left(\frac{\beta|m|}{2}\right)N_f. \qquad (22)$$

This temperature dependent shift $\delta\kappa$ is precisely the result obtained [5–8] in fermion and/or Chern-Simons theory in $2+1$ dimensions. There is an obvious physical interpretation of the correction in (22): this shift is just the finite temperature expectation value (7) of the fermion charge. In the three dimensional theory the corresponding correction to κ can be interpreted as the induced charge density per magnetic field. In the high temperature limit, the expectation value of the fermion charge goes to zero as the energy gap between the excited states and the ground state is negligible in this limit.

Attempting to identify the shifted coefficient in (22) as a new Chern-Simons coefficient, which should take discrete values in its own right, leads immediately to the difficulties discussed in the introduction and in Refs. [6,9]. However, as is very clear from our exactly solvable model, such an identification is *incorrect* at finite temperature because it ignores the higher terms in the perturbative expansion of the effective action. At zero temperature it happens to be correct to make this identification because only the first term in the expansion (14) survives in the zero temperature limit. Indeed, we see from the exact finite T effective action (11) that the entire effective action

has a well-defined behavior under a large gauge transformation, *independent of the temperature,* even though at any given finite order of a perturbation expansion there is a temperature dependence. Under a large gauge transformation, $a \to a + 2\pi N$, the effective action (11) is shifted by $N_f N \pi i$, which is exactly the same behavior as at zero temperature—see (12) and (13). This global flavor anomaly may be avoided, independent of the temperature, by considering an even number of fermion flavors. However, if the effective action is computed to any finite order in perturbation theory, its transformation under a large gauge transformation is complicated and temperature dependent.

This simple model implies that discussion of the gauge invariance of finite temperature effective actions and induced Chern-Simons terms in higher dimensions requires, at the very least, consideration of the full perturbation series. Conversely, no sensible conclusion may be drawn by considering only the first term in the expansion, as previous work has attempted to do. Our work suggests that once we remove the global flavor anomaly we expect the entire finite temperature effective action to be invariant under large gauge transformations.

An interesting feature of this model is that the finite temperature effective action is not an extensive quantity in Euclidean time. While we expect an effective action to be an extensive quantity in space, there is no reason why it should be so in Euclidean time. We expect that in the three dimensional calculation of the finite temperature effective action we could expand in the spatial derivatives and spatial components of the gauge field, but we would need to keep *all* terms in time integrations and time derivatives and in A_0. This requirement explains why the standard argument for gauge invariance of just the Chern-Simons-like term in the effective action, based on an arbitrary scaling of large gauge transformations, works at zero T but fails at nonzero T [6]. Nevertheless, the leading order term in a spatial expansion should itself be invariant under large gauge transformations. It would be interesting to find the exact expression for this effective action and its physical meaning.

Finally, the Chern-Simons quantum mechanics model considered here may be generalized to incorporate also bosonic degrees of freedom, with conserved U(1) charge Q_B. Then the Gauss law constraint becomes $\kappa + Q_F + Q_B = 0$, which leads to interesting superselection sectors of integer total charge. In addition, one can introduce Yukawa couplings between the bosonic and fermionic fields, and supersymmetrize the system. We believe further investigation in this direction will be rewarding.

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